Announcements

1) Problem set 1 up on CTools under "Assignments".

2) Piazza App (free)

Theorem: If $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$, then f' satisfies the conclusion of the Intermediate Value Theorem.

proof: Suppose that $f'(a)\le f'(b)$, w.l.o.g.

Take c, $f'(a)\le c\le f'(b)$.

Define

$$g(x) = f(x) - cx.$$

What we want: $\exists$

$d \in [a,b]$, $g'(d) = 0$.

Why? $g'(x) = f'(x) - c$,

so if $g'(d) = 0$, $f'(d) = c$.

Suppose $\exists$ $e, h \in [a,b]$,

$e \neq h$, $g(e) = g(h)$.

Then by the mean-value theorem, $\exists$ $d \in [e,h]$,

$$g'(d) = \frac{g(e) - g(h)}{e - h} = 0,$$

and so we are done.

Suppose, by contradiction, that g is injective.

Since g is continuous, g is then always increasing or always decreasing.

But $g'(a) = f'(a) - c \leq 0$ and $g'(b) = f'(b) - c \geq 0$.

If $f'(a) < c < f'(b)$, then $g'(a) < 0$ and $g'(b) > 0$, contradiction since then g is both increasing and decreasing on $[a,b]$. Hence, f' satisfies the conclusion of IVT. $\square$

Chapter 7: Integration

Basic question: to what extent are discontinuous functions $f: [a,b] \to \mathbb{R}$ (Riemann) integrable?

Defining the Riemann Integral

Definition: (partition) A partition of a closed interval $[a,b]$ is a finite collection of points $x_0, x_1, \dots, x_n$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

We denote the partition by P.

Definition: (upper sum) Let $f: [a,b] \to \mathbb{R}$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. The upper sum of f with respect to P is

$$\boxed{U(f,P) = \sum_{i=1}^{n} M_i \,(x_i - x_{i-1})}$$

where $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$

Definition: (lower sum) Let $f : [a,b] \to \mathbb{R}$ and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. The **lower sum** of f with respect to P is

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

where $m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$

Scholium: $U(f,P) \geq L(f,P)$

Proof:

$$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$\geq \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$= L(f,P)$$

Since $x_i > x_{i-1}$ and

$M_i \geq m_i$. $\square$

Definition: (Riemann Integral)

Define, for $f : [a,b] \to \mathbb{R}$

$$U(f) = \inf \left\{ U(f,P) \mid P \text{ partitions } [a,b] \right\}$$

$$L(f) = \sup \left\{ L(f,P) \mid P \text{ partitions } [a,b] \right\}.$$

We say f is Riemann Integrable if

$$\boxed{U(f) = L(f)}$$

Example 1: $f:[0,1]\to\mathbb{R}$

$$f(x)=\begin{cases}1, & x\notin\mathbb{Q}\\ 0, & x\in\mathbb{Q}\end{cases}$$

No matter what partition P you choose,

$$U(f,P)=\sum_{i=1}^{n} M_i\,(x_i-x_{i-1})$$

$$=\sum_{i=1}^{n}(x_i-x_{i-1})=1$$

Since the irrationals are dense in $\mathbb{R}$, since $M_i = 1 \quad \forall\, 1 \le i \le n$.

Similarly, by density of $\mathbb{Q}$ in $\mathbb{R}$,

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$= 0.$$

Hence $U(f) = 1 > 0 = L(f)$,

so f is NOT Riemann integrable. but f is nowhere-continuous...

Definition: (refinement of a partition)

A partition Q of $[a,b]$ is a refinement of a partition P of $[a,b]$ if

$$P \subset Q.$$

Lemma: (refinements) If Q is a refinement of P on $[a,b]$, then

$$L(f,Q) \geq L(f,P)$$

$$U(f,Q) \leq U(f,P)$$

proof: By induction. Let $Q = P \cup \{y\}$ where $y \in [a,b]$, $y \notin P$.

Let $P = \{x_0, x_1, \dots, x_n\}$.

Then $\exists\ k$, $1 \le k \le n$,

$y \in [x_{k-1}, x_k]$.

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$= \sum_{i=1}^{k-1} m_i (x_i - x_{i-1}) + m_k (x_k - x_{k-1})$$

$$+ \sum_{i=k+1}^{n} m_i (x_i - x_{i-1})$$

Then $L(f, Q)$ will be the same except for the $m_k(x_k - x_{k-1})$ term. We would replace it by $c(y - x_{k-1}) + d(x_k - y)$

where $c = \inf_{x \in [x_{k-1}, y]} f(x)$, $d = \inf_{x \in [y, x_k]} f(x)$.

We have $c(y - x_{k-1}) + d(x_k - y)$

$$\geq m_k(x_k - x_{k-1}).$$

Similarly,

$$U(f, Q) \le U(f, P).$$

Finish the proof using induction. $\square$